# Minimal surfaces in $AdS$ and the eight-gluon scattering amplitude at strong coupling



Luis F. Alday and Juan Maldacena

*School of Natural Sciences, Institute for Advanced Study*
*Princeton, NJ 08540, USA*

In this note we consider minimal surfaces in three dimensional anti-de Sitter space that end at the $AdS$ boundary on a polygon given by a sequence of null segments. The problem can be reduced to a certain generalized Sinh-Gordon equation and to $SU(2)$ Hitchin equations. The mathematical problem to be solved arises also in the context of the moduli space of certain three dimensional supersymmetric theories. We can use explicit results available in the literature in order to find the explicit answer for the area of a surface that ends on a eight-sided null Wilson loop. Via the gauge/gravity duality this can also be interpreted as a certain eight-gluon scattering amplitude at strong coupling for a special kinematic configuration.

## 1. Introduction

Recently there has been some interest in Wilson loops that consist of a sequence of light-like segments. These are interesting for several reasons. First, they are a simple subclass of Wilson loops which depend on a finite number of parameters, the positions of the cusps. Second, they are Lorentzian objects with no obvious Euclidean counterpart. Finally, it was shown that they are connected to scattering amplitudes in gauge theories [1,2,3,4,5], for a review see [6].

In this note we study these Wilson loops at strong coupling by using the gauge/string duality. One is then led to compute the area of minimal surfaces in $AdS$ [7][8]. We consider a special class of null Wilson loops which can be embedded in a two dimensional subspace, which we can take as an $R^{1,1}$ subspace of the boundary of $AdS$. For these loops, the string worldsheet lives in an $AdS_3$ subspace of the full $AdS_d$ space, $d \geq 3$.

In order to analyze the problem one can use a Pohlmeyer type reduction [9,10,11,12]. This maps the problem of strings moving in $AdS_3$ to a problem involving a single field $\alpha$ which obeys a generalized Sinh-Gordon equation.

The same mathematical problem appears in the study of $SU(2)$ Hitchin equations [13]. Interestingly, these Hitchin equations also appear in the study of the supersymmetric vacua of certain gauge theories [14,15]. This connection is specially useful because the authors of [14,15] have studied this problem, exploiting its integrability, and have proposed a method for finding the answer. For the simplest case, the metric in moduli space for the corresponding field theory problem is known [16,17,14,15]. These results can be used to compute the area for the simplest non-trivial case which is the eight sided Wilson loop.

This note is organized as follows. In section 2 we describe the interplay between classical strings on $AdS_3$ and the generalized Sinh-Gordon model. In section 3 we describe several features of the solutions at hand and in section 4 we give the full answer for the case of the null Wilson loop with eight sides.

Note: A much more detailed exposition of the material reported here will be presented in a forthcoming publication [18].

## 2. Sinh-Gordon model from strings on $AdS_3$

Classical strings in $AdS$ spaces can be described by a reduced model which depends only on physical degrees of freedom. In terms of embedding coordinates, $Y^\mu$ in $R^{2,2}$, with $Y^2 = -1$, the conformal gauge equations of motion and Virasoro constraints are

$$\partial\bar{\partial}\vec{Y} - (\partial\vec{Y}.\bar{\partial}\vec{Y})\vec{Y} = 0 \ , \qquad \partial\vec{Y}.\partial\vec{Y} = \bar{\partial}\vec{Y}.\bar{\partial}\vec{Y} = 0 \tag{2.1}$$

where we parametrize the world-sheet in terms of complex variables $z$ and $\bar{z}$. For the case of $AdS_3$ the above system can be reduced to the generalized sinh-Gordon model [9,10,11,12]. We define

$$\begin{aligned} e^{2\alpha(z,\bar{z})} &= \frac{1}{2}\partial\vec{Y}.\bar{\partial}\vec{Y}, \qquad N_a = \frac{e^{-2\alpha}}{2}\epsilon_{abcd}Y^b\partial Y^c\bar{\partial}Y^d \ , \\ p &= -\frac{1}{2}\vec{N}.\partial^2\vec{Y} \ , \qquad \bar{p} = \frac{1}{2}\vec{N}.\bar{\partial}^2\vec{Y} \end{aligned} \tag{2.2}$$

Then, as a consequence of (2.1) it can be shown that $p = p(z)$ is a holomorphic function and that $\alpha(z, \bar{z})$ satisfies the generalized Sinh-Gordon equation

$$\partial\bar{\partial}\alpha(z,\bar{z}) - e^{2\alpha(z,\bar{z})} + |p(z)|^2 e^{-2\alpha(z,\bar{z})} = 0 \tag{2.3}$$

The area of the world-sheet is simply given by

$$A = 4\int d^2 z e^{2\alpha} \tag{2.4}$$

For solutions relevant to this note this area is divergent and we will need to regularize it.

Given a solution of the generalized sinh-Gordon model, one can reconstruct a classical string world-sheet in $AdS_3$ by solving two auxiliary linear problems (which we will denote as left and right)

$$\begin{aligned} \partial\psi^L_{\alpha,a} + (B^L_z)_\alpha^{\ \beta}\psi^L_{\beta,a} &= 0, \qquad \bar{\partial}\psi^L_{\alpha,a} + (B^L_{\bar{z}})_\alpha^{\ \beta}\psi^L_{\beta,a} = 0 \\ \partial\psi^R_{\dot{\alpha},\dot{a}} + (B^R_z)_{\dot{\alpha}}^{\ \dot{\beta}}\psi^R_{\dot{\beta},\dot{a}} &= 0, \qquad \bar{\partial}\psi^R_{\dot{\alpha},\dot{a}} + (B^R_z)_{\dot{\alpha}}^{\ \dot{\beta}}\psi^R_{\dot{\beta},\dot{a}} = 0 \end{aligned} \tag{2.5}$$

where the $SL(2)$ left and right flat connections are given by

$$\begin{aligned} B^L_z &= \begin{pmatrix} \frac{1}{2}\partial\alpha & -e^{\alpha} \\ -e^{-\alpha}p(z) & -\frac{1}{2}\partial\alpha \end{pmatrix}, \qquad B^L_{\bar{z}} = \begin{pmatrix} -\frac{1}{2}\bar{\partial}\alpha & -e^{-\alpha}\bar{p}(\bar{z}) \\ -e^{\alpha} & \frac{1}{2}\bar{\partial}\alpha \end{pmatrix} \\ B^R_z &= \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha}p(z) \\ -e^{\alpha} & \frac{1}{2}\partial\alpha \end{pmatrix}, \qquad B^R_{\bar{z}} = \begin{pmatrix} \frac{1}{2}\bar{\partial}\alpha & -e^{\alpha} \\ e^{-\alpha}\bar{p}(\bar{z}) & -\frac{1}{2}\bar{\partial}\alpha \end{pmatrix} \end{aligned} \tag{2.6}$$

Internal $SL(2)_L \times SL(2)_R$ indices $\alpha, \dot{\alpha}$ denote rows and columns of these connections, while the indices $a$ and $\dot{a}$ denote the two different linearly independent solutions of each linear problem (2.5) . The space-time isometry group $SO(2,2) = SL(2) \times SL(2)$ acts on these indices. We require that the two pairs of solutions obey the normalization condition

$$\psi^L_a \wedge \psi^L_b \equiv \epsilon^{\beta\alpha}\psi^L_{\alpha,a}\psi^L_{\beta,b} = \epsilon_{ab}, \qquad \psi^R_{\dot{a}} \wedge \psi^R_{\dot{b}} = \epsilon_{\dot{a}\dot{b}} \tag{2.7}$$

Once a pair of solutions has been found, the explicit form of the space-time coordinates $Y_{a\dot{a}}(z,\bar{z})$ is given by a particular bilinear combination of the left and right solutions

$$Y_{a\dot{a}} = \begin{pmatrix} Y_{-1}+Y_2 & Y_1-Y_0 \\ Y_1+Y_0 & Y_{-1}-Y_2 \end{pmatrix}_{a,\dot{a}} = \psi^L_{\alpha,a} M_1^{\alpha\dot{\beta}} \psi^R_{\dot{\beta},\dot{a}} \ , \qquad M_1^{\alpha\dot{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.8}$$

It turns out that the left connection $B^L$ can be promoted to a family of flat connections by introducing a spectral parameter

$$B_z = A_z + \Phi_z \to B_z(\zeta) = A_z + \frac{1}{\zeta}\Phi_z, \qquad B_{\bar{z}} = A_{\bar{z}} + \Phi_{\bar{z}} \to B_{\bar{z}}(\zeta) = A_{\bar{z}} + \zeta\Phi_{\bar{z}} \tag{2.9}$$

where we have decomposed the connection into its diagonal part $A_z$ and off diagonal part $\Phi_z$. Actually, both, left and right connections can be simply obtained (up to a constant gauge transformation) from $B(\zeta)$ by setting $\zeta = 1$ or $\zeta = i$ respectively. The zero curvature condition for $B(\zeta)$ can be rephrased as

$$\begin{aligned} D_{\bar{z}}\Phi_z = D_z\Phi_{\bar{z}} = 0, \qquad & F_{z\bar{z}} + [\Phi_z, \Phi_{\bar{z}}] = 0 \\ D_\mu\Phi = \partial_\mu\Phi + [A_\mu, \Phi], \qquad & F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \end{aligned} \tag{2.10}$$

These are the Hitchin equations, which arise by dimensional reduction of the four dimensional self-duality condition (instanton equations) to two dimensions. $A$ has the interpretation of a gauge connection in two dimensions and $\Phi$ is a Higgs field. In our case we have the Hitchin equations for $SU(2)$.

After we compute the area we can relate these results to results for Wilson loops or amplitudes in $\mathcal{N}=4$ super Yang Mills as

$$\langle W \rangle \sim \text{Amplitude} \sim e^{-\frac{R^2}{2\pi\alpha'}(\text{Area})} \ , \qquad \frac{R^2}{\alpha'} = \sqrt{\lambda} = \sqrt{g^2 N} \tag{2.11}$$

where the area is computed by setting the $AdS$ radius to one and we have embedded $AdS_3$ appropriately in $AdS_5$. For other field theories with gravity duals we should use the corresponding expression for $R^2/\alpha'$.

## 3. Finding the surface and computing its area

Since $p(z)$ is a holomorphic function we can simplify the generalized sinh-Gordon equation by defining a new variable $dw = \sqrt{p(z)}dz$. In the $w-$plane the equation takes the form

$$\partial_w \bar{\partial}_{\bar{w}} \hat{\alpha} - e^{2\hat{\alpha}} + e^{-2\hat{\alpha}} = 0 \ , \qquad \hat{\alpha} \equiv \alpha - \frac{1}{4}\log p\bar{p} \tag{3.1}$$

The expression for the area (2.4) becomes $A = 4\int d^2 w e^{2\hat{\alpha}}$. Note that $w$ has branch cuts where $p$ has zeros. So, we locally simplify the equation but we complicate the space on which the equation is defined.

Let us first write the solution for the four sided polygon obtained in [19,1]. In this case $p = 1$, $w = z$ and $\alpha = 0$. The connections (2.6) are constant. Up to a constant gauge transformation the two solutions can be chosen as

$$\psi_+^L = \begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix} , \qquad \psi_-^L = \begin{pmatrix} 0 \\ e^{-(w+\bar{w})} \end{pmatrix} \tag{3.2}$$

The $\psi_\pm^R$ are the same but with $w + \bar{w} \to \frac{w-\bar{w}}{i}$. Some of these solutions diverge when $|z| \to \infty$. This implies that the AdS embedding coordinates (2.8) are diverging, which means that we are going to the $AdS$ boundary. Different components of the $Y$ coordinates in (2.8) are different combinations of the solutions $\psi_\pm^{L,R}$. The solutions that diverges determine a point on the $AdS$ boundary. This point depends on whether the solution that diverges is $\psi_+$ or $\psi_-$. Thus, in each of the four quadrants of the $w$ plane we approach a different point on the $AdS$ boundary. Each of these points is a cusp. When we change quadrants the solutions change dominance only for the left problem or only for the right problem. This implies that two consecutive cusps are separated by a null line.

Consider now the case in which $p(z)$ is a generic polynomial of degree $n-2$, $p \sim z^{n-2} + \cdots$. In this case $w \sim z^{n/2}$ for large $z$. As we go once around the $z$ plane we go around $n/2$ times in the $w$ plane. Since we expect that the solution near each cusp is similar to the solution for the four sided polygon, we demand that $\hat{\alpha} \to 0$ as $|z|$ goes to infinity. As a result, the solutions for large $w$ become approximately as in (3.2). The problem displays the Stokes phenomenon. Namely, an exact solution is given in terms of a linear combination of each of the two solutions (3.2), with coefficients that change as we move between Stokes sectors. For instance, consider the left problem in the large $|w|$ region, such as $Im(w) < 0$. Within that region (3.2) is a good approximate solution. Let us consider now what happens as we cross the line where $w$ is real and positive, with very large $|w|$. The solution that decreases as $Re(w) \to +\infty$ is accurately given by (3.2) and it is the same on both sides of the line. On the other hand, the large solution will have a jump in its small solution component. More precisely, we choose a basis of solutions which has the asymptotic form in (3.2) in one Stokes sector. We denote these two exact solutions as $\psi_\pm^{\rm before}$. After we cross the Stokes line, we enter into a new Stokes sector. We can now choose another basis of solutions which has the asymptotic form in (3.2) in this

new sector. We denote these two exact solutions as $\psi_\pm^{\rm after}$. These two sets of solutions should be related by a simple linear transformation. In fact we have

$$\psi_a^{\rm before} = S_a^{\ b}\, \psi_b^{\rm after} \ , \qquad\qquad S(\gamma) = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \tag{3.3}$$

The Stokes matrix acts on the target space $SL(2)$ index, $a = \pm$, of the solutions. In other words $\psi_\pm^{\rm before}$ has a new asymptotic expression in the new sector. A pair of exact solutions has an approximate expression in each sector which is given by a linear combination of the two solutions in (3.2). These coefficients also change by multiplication of the matrix in (3.3) as we change sectors.

The right problem will display a similar phenomenon. The discontinuities will be then characterized by Stokes parameters $\gamma_i^L$ and $\gamma_i^R$, where $i = 1, .., n$ runs over the Stokes lines. In general, the value of the $\gamma' s$ depends on the full exact solution and cannot be computed purely at large $|w|$.

The Stokes lines for the left linear problem (2.6) are at $Re(w) = 0$ and the ones for the right problem are at $Im(w) = 0$. In addition, within each Stokes sector the solutions in (3.2) exchange dominance at the anti-Stokes lines, which for the left problem are at $Im(w) = 0$ and for the right problem are at $Re(w) = 0$. In conclusion, in each quadrant of the $w$ plane the solution diverges and it approaches a point on the $AdS_3$ boundary, the location of a cusp. As we change from one quadrant to the next we encounter an anti-Stokes line of one of the two problems, say the left problem, and now a different solution is diverging, leading to a different point on the $AdS$ boundary. At this anti-Stokes line the right problem continues to have the same dominant solution. This implies that the two points corresponding to two consecutive quadrants are related by only a left target space $SL(2)$ transformation and are thus lightlike separated. In conclusion, a polynomial $p$ of degree $n-2$ leads to $2n$ different quadrants in the $w$ plane and thus $2n$ cusps. We can form $2(n-3)$ different spacetime cross ratios out of these points. This number coincides with the number of non-trivial real parameters in the coefficients of a polynomial of degree $n-2$, $p = z^{n-2} + c_{n-4} z^{n-4} + \cdots + c_0$, where we used scalings and translations to bring the polynomial to this form. The first case where we can form cross ratios is $n = 4$ which corresponds to the octagon. For Wilson loops made of null segments in $R^{1,3}$ the first non-trivial case is the six sided polygon.

One can write an elegant formula for the left spacetime cross ratios purely in terms of the solutions of the left linear problem in (2.6). Choosing Poincare coordinates for $AdS_3$,

$ds^2 = (dx^+ dx^- + dr^2)/r^2$, we can write the spacetime cross ratios for four points $x_i^+$, not necessarily consecutive, as

$$\frac{x_{12}^+ x_{34}^+}{x_{13}^+ x_{24}^+} = \frac{s_1^L \wedge s_2^L \, s_3^L \wedge s_4^L}{s_1^L \wedge s_3^L \, s_2^L \wedge s_4^L} \equiv \chi(\zeta = 1) \tag{3.4}$$

where $s_i$ is the small solution at the cusp $i$. The $s_i$ are well defined up to a rescaling, which cancels in (3.4). We have similar expressions for the right problem, which can be obtained by introducing the spectral parameter and setting $\zeta = i$. The expressions in the right hand side are the cross ratios introduced in [15,20].

A very similar mathematical problem was considered by Gaiotto, Moore and Neitzke in a very different context [14]. Their motivation was the study of the Hyperkahler moduli space of certain three dimensional gauge theories with $\mathcal{N} = 4$ susy. Those theories can arise from wrapping $D4-$branes on Riemann surfaces. The classical Higgs branch moduli space of vacua of these theories is given by the moduli space of the Hitchin equations on the corresponding Riemann surface. The moduli space is parametrized by the coefficients of the polynomial $p = \prod_{i=1}^{n-2}(z - z_i)$, $\sum_i z_i = 0$. The authors of [14] have studied the analytic structure of the cross-ratios (3.4)as a function of $\zeta$ and have written a Riemann-Hilbert problem whose solutions determine the metric in moduli space $g_{z_i \bar{z}_j}$. By computing the Kahler potential that leads to this metric we can write an expression for the area as

$$A \sim \sum_i (z_i \partial_{z_i} + \bar{z}_i \partial_{\bar{z}_i}) K \ , \qquad \partial_{z_i} \partial_{\bar{z}_j} K = g_{z_i \bar{z}_j} \tag{3.5}$$

In principle this should determine the full solution of the problem. The metric is known explicitly for the case that corresponds to four dimensional $\mathcal{N} = 2$ theory with a single hypermultiplet compactified on a circle, which have been considered in [16,17]. In this case the metric is a multi-Taub-Nut metric. This corresponds to a case with only one complex modulus, which leads to the octagon in our case [1].

## 4. The octagon

In the case of the octagon we have $n = 4$ and the polynomial is $p \sim z^2 - m$. As $\sqrt{p(z)} = z - \frac{m}{2z} + \ldots$, we cover the $w-$plane twice, but, in addition we undergo a shift

[1] We thank Davide Gaiotto for pointing out this relation.

$w \to w + w_s$, with $w_s = -i\pi m$, as we go around twice. We can then say that the $w-$plane is missing a sliver of "width" $w_s$. The fact that the information about $m$ survives at large $|w|$ has the nice consequence that it allows us to compute the space-time cross-ratios exactly as a function of $m$

$$\begin{aligned} \chi^+ \equiv & e^{w_s + \bar{w}_s} = e^{\pi(\frac{m-\bar{m}}{i})} = \frac{(x_4^+ - x_1^+)(x_3^+ - x_2^+)}{(x_4^+ - x_3^+)(x_2^+ - x_1^+)} \\ \chi^- \equiv & e^{\frac{w_s - \bar{w}_s}{i}} = e^{-\pi(m+\bar{m})} = \frac{(x_4^- - x_1^-)(x_3^- - x_2^-)}{(x_4^- - x_3^-)(x_2^- - x_1^-)} \end{aligned} \tag{4.1}$$

We can now apply the known explicit formulas for the metric to compute the area. The area (2.4) is divergent and needs to be regularized. We consider a physical regularization which corresponds to placing a cut-off on the radial $AdS_3$ direction, $r \geq \mu$. This cut-off renders the area finite since it does not allow arbitrarily large values of $|z|$ or $|w|$.

In order to extract the dependence on the regulator it is convenient to write the area as the sum of two pieces

$$\begin{aligned} A = & 4\int d^2z(e^{2\alpha} - \sqrt{p\bar{p}}) + 4\int_{r(z,\bar{z})\geq\mu} d^2z\sqrt{p\bar{p}} = A_{Sinh} + 4\int_\Sigma d^2w \\ & \text{with} \quad A_{Sinh} = 4\int d^2z(e^{2\alpha} - \sqrt{p\bar{p}}) = 4\int d^2w(e^{2\hat{\alpha}} - 1) \end{aligned} \tag{4.2}$$

In order to regulate the second term in (4.2) we need to know the asymptotic behavior of the radial coordinate $r(z,\bar{z})$. This appears to require a full explicit solution to the problem. However, we also know that the asymptotic form of the solution also determines the positions of the cusps, which in turn determine the kinematic invariants such as the distance between cusps. Indeed, most of the dependence on the explicit solution $r(z,\bar{z})$ can be reexpressed in terms of the kinematic invariants. We then find that the second piece in (4.2) can be written as the sum of several terms

$$4\int_\Sigma d^2w = A_{div} + A_{BDS} - \frac{1}{2}\log(1+\chi^-)\log(1+\frac{1}{\chi^+}) - \frac{\pi}{2}(m+\bar{m})\log\gamma_1^L - \frac{\pi}{2}\frac{m-\bar{m}}{i}\log\gamma_1^R \tag{4.3}$$

The first term is the well known divergent piece with the appropriate infra red behavior. $A_{BDS}$ is the function that appears at one loop in perturbation theory [21], which solves the anomalous conformal ward identities [2],

$$A_{BDS} = -\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i,i-1}^{n}\log\frac{x_j^+ - x_i^+}{x_{j+1}^+ - x_i^+}\log\frac{x_j^- - x_{i-1}^-}{x_j^- - x_i^-} \tag{4.4}$$

The last term in (4.3) depends explicitly on the Stokes parameters and is a consequence of the sliver missing in the $w-$plane. The explicit values of the Stokes parameters are given in [14]

$$\begin{aligned} \log \gamma_1(\zeta) &= \frac{e^{-i\phi}\zeta}{\pi} \int_{-\infty}^{\infty} dt \frac{e^t}{e^{2t} + \zeta^2 e^{-i2\phi}} \log\left(1 + e^{-2|m|\pi \cosh t}\right) \\ \log \gamma_1^L = \log \gamma_1(\zeta = 1) \ , & \qquad \log \gamma_1^R = \log \gamma_1(\zeta = i) \ , \qquad m = |m| e^{i\phi} \end{aligned} \tag{4.5}$$

Finally, $A_{Sinh}$ in (4.2) can be computed by considering the metric in moduli space for this problem [16,17,14]

$$\partial_m \partial_{\bar{m}} K = g_{m\bar{m}} \sim \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{|m|^2 + (n+1/2)^2}} + \text{const} \tag{4.6}$$

and using (3.5). Putting all this together we obtain the final answer for the null octagon

$$\begin{aligned} A =& A_{div} + A_{BDS} + R \\ R =& -\frac{1}{2} \log(1+\chi^-) \log(1 + \frac{1}{\chi^+}) + \frac{7\pi}{6} + \int_{-\infty}^{\infty} dt \frac{|m| \sinh t}{\tanh(2t + 2i\phi)} \log\left(1 + e^{-2\pi |m| \cosh t}\right) \end{aligned} \tag{4.7}$$

we have written the final answer in terms of the $BDS$ expression (4.4) plus a remainder function, $R$, which is a function of the cross ratios. For the special kinematical configuration we have considered in this note, this remainder functions depends on two cross ratios (out of a total of nine for the generic case in four dimensions). One way to think of this kinematic configuration is in terms of four left moving gluons $p_i^+$ and four right moving ones with $p_i^-$, $i = 1, \cdots, 4$ in an $R^{1,1}$ subspace. Then $x_{i+1,i}^{\pm} = p_i^{\pm}$. This expression has the correct behavior for all the limiting cases and the correct analytic structure. It is analyzed in much greater detail in [18]. Two loop perturbative expressions for the Wilson null polygons were given in [22].

**Acknowledgments**

We are very grateful to Davide Gaiotto, for a large number of important suggestions, for his detailed explanations of [14] and for giving us a draft of [15]. We also thank N. Arkani-Hamed, V. Kazakov, P. Vieira for discussions.

This work was supported in part by U.S. Department of Energy grant #DE-FG02-90ER40542.

## References


[1] L. F. Alday and J. M. Maldacena, JHEP **0706**, 064 (2007) [arXiv:0705.0303 [hep-th]].

[2] J. M. Drummond, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B **795**, 385 (2008) [arXiv:0707.0243 [hep-th]].

[3] A. Brandhuber, P. Heslop and G. Travaglini, Nucl. Phys. B **794**, 231 (2008) [arXiv:0707.1153 [hep-th]].

[4] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. D **78**, 045007 (2008) [arXiv:0803.1465 [hep-th]].

[5] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, arXiv:0803.1466 [hep-th].

[6] L. F. Alday and R. Roiban, Phys. Rept. **468**, 153 (2008) [arXiv:0807.1889 [hep-th]].

[7] J. M. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998) [arXiv:hep-th/9803002].

[8] S. J. Rey and J. T. Yee, Eur. Phys. J. C **22**, 379 (2001) [arXiv:hep-th/9803001].

[9] K. Pohlmeyer, Commun. Math. Phys. **46**, 207 (1976).

[10] H. J. De Vega and N. G. Sanchez, Phys. Rev. D **47**, 3394 (1993).

[11] A. Jevicki, K. Jin, C. Kalousios and A. Volovich, arXiv:0712.1193 [hep-th].

[12] A. Jevicki and K. Jin, arXiv:0903.3389 [hep-th].

[13] N. J. Hitchin, Proc. Lond. Math. Soc. **55**, 59 (1987).

[14] D. Gaiotto, G. W. Moore and A. Neitzke, arXiv:0807.4723 [hep-th].

[15] D. Gaiotto, G. W. Moore and A. Neitzke, to appear

[16] H. Ooguri and C. Vafa, Phys. Rev. Lett. **77**, 3296 (1996) [arXiv:hep-th/9608079].

[17] N. Seiberg and S. H. Shenker, Phys. Lett. B **388**, 521 (1996) [arXiv:hep-th/9608086].

[18] L. F. Alday and J. Maldacena, To appear.

[19] M. Kruczenski, JHEP **0212**, 024 (2002) [arXiv:hep-th/0210115].

[20] V.V. Fock, A.B. Goncharov, arXiv:math/0311149

[21] Z. Bern, L. J. Dixon and V. A. Smirnov, Phys. Rev. D **72**, 085001 (2005) [arXiv:hep-th/0505205].

[22] C. Anastasiou, A. Brandhuber, P. Heslop, V. V. Khoze, B. Spence and G. Travaglini, arXiv:0902.2245 [hep-th].